SPP 2410 Hyperbolic Balance Laws in Fluid Mechanics: Complexity, Scales, Randomness (CoScaRa)

Compressible Euler equations with transport noise

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Motivation

The motivation to study transport noise in fluid mechanics is twofold:

- Holm et al. derived in [7] models with transport noise in fluid dynamics from a physical perspective to model turbulent effects.
- Transport noise can have regularisation effects as demonstrated in [8] for the transport equation and very recently in [9] for the 3D incompressible Navier–Stokes equations.

The Mathematical Problem

We consider the complete stochastic Euler equations, describing the flow of a general compressible and heat-dependent fluid in a bounded domain $\mathcal{O} \subset \mathbb{R}^n$, $n = 1, 2, 3$:

$$\begin{cases} \partial_t \mathbf{m} + \operatorname{div}\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right) = -\nabla p + \dot{\boldsymbol{\eta}}, \\ \partial_t \varrho + \operatorname{div}(\mathbf{m}) = \dot{\xi}. \\ \partial_t s + \operatorname{div}\left(\frac{s\mathbf{m}}{\varrho}\right) = \dot{\chi}. \end{cases} \tag{1}$$

The unknowns are the momentum $\mathbf{m}$, the density ϱ and the specific entropy s (or the absolute temperature ϑ). We suppose the the the Boyle-Mariotte thermal equation law $p = \varrho\vartheta$.

Momentum-type transport noise. We speak about transport noise if

$$\dot{\boldsymbol{\eta}} = \sum_{k=1}^{K} \operatorname{div}(\boldsymbol{\sigma}_k \otimes \mathbf{m}) \circ \frac{dW_k}{dt}, \quad \dot{\xi} = \sum_{k=1}^{K} \operatorname{div}(\varrho\boldsymbol{\sigma}_k) \circ \frac{dW_k}{dt}, \tag{2}$$

with given vector fields $\boldsymbol{\sigma}_k$ and stochastic differentials with respect to independent Wiener process W_k in the Stratonovich sense.

Velocity-type transport noise. As an alternative to the transport noise described above we consider the noise

$$\dot{\boldsymbol{\eta}} = \sum_{k=1}^{K} \operatorname{div}\left(\boldsymbol{\sigma}_k \otimes \frac{\mathbf{m}}{\varrho}\right) \circ \frac{dW_k}{dt}, \quad \dot{\xi} = 0 \quad \dot{\chi} = ?. \tag{3}$$

Preliminary Work

Stochastic forcing. We speak of stochastic forcing if

$$\dot{\boldsymbol{\eta}} = \Phi(\varrho, \mathbf{m}, s)\frac{dW}{dt}, \quad \dot{\xi} = 0, \quad \dot{\chi} = 0,$$

with a (possibly infinite-dimensional) Wiener process W and an operator Φ with appropriate growth assumptions. Measure-valued martingale solutions to (1) with stochastic forcing were constructed in [10].

Much more is known in the viscous case: The Navier–Stokes–Fourier equations with stochastic forcing have been studied in [2, 3, 11], where the existence of (analytically) weak martingale solutions is shown. They built on the systematic study of the isentropic problem from [4].

Momentum-type transport noise (2). In the isentropic case (where only the first two equation in (1) are considered and $p = p(\varrho) = a\varrho^\gamma$ with $a > 0$ and $\gamma > 1$), considered in the first funding period, the existence of measure-valued martingale solutions is shown in [1], along with their weak-strong uniqueness and the existence of a Markov selection. This is based on the studies on the compressible Navier–Stokes system with the same transport noise from [6].

Objectives

We aim to analyse the complete compressible Euler equations (1) with transport noise as in (2) or (3).

Momentum-type transport (2)

(WP1) Before we come to the study of the Euler equations (1), we will analyse the corresponding Navier–Stokes equations (where $(1)_1$ and $(1)_3$ contain diffusive terms).

(WP2) The solutions to the Euler equations will be constructed as inviscid limit. We aim to prove the existence of measure-valued martingale solutions to the Euler equations (1). Eventually, we will investigate the weak-strong uniqueness property and the existence of Markov selections.

Velocity-type transport (3)

(WP3) As in (WP1) we will start with the corresponding Navier–Stokes system (where $(1)_1$ and $(1)_3$ contain diffusive terms). In addition to the analysis of weak martingale solutions, we plan to investigate the possibility of enhanced dissipation by a carefully chosen noise.

(WP4) As in (WP2) we aim to prove the existence of measure-valued martingale solutions. Finding a suitable definition of a solution, which allows for the weak-strong uniqueness property, is part of the problem. A final goal is the enhanced dissipation property.

The Project's Research in the Context of SPP 2410

The complete Euler system (1) is an iconic example of a hyperbolic conservation law which is ubiquitous in many applications in physics and engineering. In turbulence theory one is often confronted with resolved large-scale, slow-varying and unresolved small-scale, fast varying components of the velocity field modeled by stochastic differential equations.

Collaborations within SPP 2410

Project-<Giesselmann-Öffner>: In the paper [5], written in the first funding period, the numerical approximation of the complete Euler equations (1) with stochastic forcing is studied. We hope that the results of this project will lead to a similar numerical study for the complete Euler equations (1) with the transport noise (2) or (3), in particular, to compare both.

References

References [1], [5] and [6] were written during the first funding period.

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