ADVANCED NUMERICAL METHODS FOR SOLVING DISPERSIVE EQUATIONS FOR WATER WAVES PROPAGATIONS

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1 Context

Many advanced fluid dynamics models feature dispersive equations or non-local parabolic equations. This is particularly the case for wave propagation models in coastal environments, such as the Boussinesq model [7] or advanced models for groundwater dynamics [6, 13]. The numerical resolution of dispersive or non-local models remains a significant scientific challenge. Most dispersive models derived as approximations to water wave models can be expressed as hyperbolic systems, where the solution is sought within a linear subspace of a weighted L^2 space. This is achieved using a Helmholtz-type decomposition, similar to approaches used in incompressible flows [11]. We refer to this mathematical framework as the **projection structure**.

Several studies have leveraged this structure, demonstrating its advantages in terms of robustness and computational efficiency. For instance, [10] introduced an entropy-satisfying numerical scheme, while [9] proposed a robust strategy for handling dry fronts or for imposing conditions on water depth, discharge, or pressure at boundaries. Furthermore, [12] presented a general framework for coupling dispersive models, preserving both the projection structure and robustness properties.

Despite these advancements, the primary obstacle to applying dispersive models to large-scale problems lies in their computational cost. The goal of this project is to design efficient numerical schemes that exploit the projection structure to overcome this limitation.

2 Purpose of the thesis

The project we propose can be summarized in three main objectives. The first two are independent but complementary, focusing on distinct strategies to accelerate the numerical resolution process. These methods will initially be tested on the KdV/BBM model [1], which is the simplest model exhibiting the projection structure. The final objective is to apply the most effective methods identified to more realistic dispersive models.

In the context of a projected hyperbolic formulation, the KdV/BBM model is expressed as follows

$$\begin{aligned} \partial_t U + \partial_x \left(F\left(U\right)\right) &= -\Psi \\ \mathcal{L}\left(U\right) &= 0 \\ \langle U, \Psi \rangle &= 0 \end{aligned} \qquad \text{where} \qquad U = \begin{pmatrix} u \\ w \end{pmatrix}, \qquad F\left(U\right) = \begin{pmatrix} \frac{u^2}{2} \\ cw \end{pmatrix} \qquad \text{and} \qquad \mathcal{L}\left(U\right) := w + \alpha \partial_x u.$$

with $c \in \mathbb{R}$ and $\alpha \in \mathbb{R}$.

The **prediction/correction schemes** appear particularly well-suited for projected hyperbolic models. These schemes consist of two steps: an explicit prediction followed by an implicit projection. For first-order schemes, the explicit prediction is given by

$$U^* = U^n - \delta_t \partial_x \left(F\left(U^n\right)\right) \tag{1}$$

and the implicit projection is defined as

$$\begin{aligned} U^{n+1} &= U^* - \delta_t \Psi^{n+1} \\ \mathcal{L}\left(U^{n+1}\right) &= 0. \end{aligned} \tag{2}$$

2.1 High-order prediction/correction schemes

Building on the literature for incompressible equations [5], projection methods offer a promising framework for developing high-order time schemes. These methods incorporate sub-time steps and include an explicit estimation of the dispersive source term Ψ^n in the prediction step (1). The implicit projection (2), which is the most computationally expensive step, is performed only at the end of each time step.

The main challenge lies in achieving high-order spatial discretization (greater than 2) for both the subspace of admissible functions and the scalar product required for the projection. Two potential approaches can address this challenge:

WENO Finite Volume Methods: This approach has the potential to produce high-order methods with a minimal number of degrees of freedom. However, it leads to linear systems with large stencils, which increase computational complexity.

Discontinuous Galerkin Discretization: This method involves a greater number of discretization points per cell but allows the projection step to be resolved locally within each cell for almost all degrees of freedom. This locality can simplify the computational process and enhance parallelization.

A second well-established strategy involves **pseudo-compressible schemes**, where the projection step is approximated by solving the time-asymptotic solution of a hyperbolic problem. The efficiency of this hyperbolic (pseudo-compressible) relaxation can be improved by fine-tuning the relaxation coefficients [8]. Additionally, we propose implementing a local detection strategy to limit computations to regions where the fictitious stationary regime has not yet been achieved, further reducing unnecessary computational overhead.

2.2 Adaptive numerical scheme based on an a priori estimator

Another approach to reducing the computational cost of the numerical strategy is to perform the projection step (2) only where and when it is necessary. To achieve this, we propose estimating regions where the predicted solution U^* significantly deviates from satisfying the constraint, using the spatial estimator $\varepsilon := \mathcal{L}(U^*)$. Where and when the estimator ε is smaller than a predefined tolerance, the solution is considered acceptable, and the projection step can be skipped.

However, since the projection step (2) is inherently non-local, its influence extends beyond the subdomain where ε exceeds the tolerance. To address this, we propose enlarging the projection subdomain by employing a thick interface coupling strategy [12]. In this approach, the parameters defining the interface thickness are dynamically determined using the estimator ε, ensuring a smooth transition and maintaining the robustness of the numerical scheme.

2.3 Application to Boussinesq-type models

The final goal of this project is to extend the proposed methods to more realistic approximate water wave models. Specifically, we aim to apply them to the fully nonlinear and weakly dispersive Green-Naghdi model [4] and the weakly nonlinear and weakly dispersive Peregrine model [14]. Additionally, more complex models incorporating a vertical profile of horizontal velocity [2, 3] are also potential candidates for investigation. These advanced models exhibit a slightly more intricate projection structure.

To keep it simple, the Green-Naghdi model on flat bottom can be written as

$$\begin{array}{c}\partial_t \begin{pmatrix} h \\ hU \end{pmatrix} + \nabla \cdot \left(F \begin{pmatrix} h \\ U \end{pmatrix} \right) = - \begin{pmatrix} 0 \\ \Psi \end{pmatrix} \\ L_h(U) = 0 \\ \langle U, \Psi \rangle_h = 0 \end{array} \quad \text{where} \quad U = \begin{pmatrix} \overline{u} \\ \overline{w} \end{pmatrix}, \quad F(h, U) = \begin{pmatrix} h\overline{u} \\ h\overline{u} \otimes \overline{u} + \frac{g}{2} h^2 \\ h\overline{w}\,\overline{u} \end{pmatrix}$$

$$\mathcal{L}_h(U) := \overline{w} + \alpha h \nabla \cdot \overline{u} \qquad \text{and} \qquad \langle V_1, V_2 \rangle_h := \int_{\mathbb{R}^d} V_1 \cdot V_2 h \,\mathrm{d}x$$

Here, the constraint $\mathcal{L}_h(U)$ and the scalar product $\langle U, \Psi \rangle_h$ depend on the evolving unknown h, adding a layer of complexity to the projection structure. The previously proposed strategies can be adapted to these models. However, special attention must be given to the discretization of the evolving variable h to ensure that the scheme maintains its high-order accuracy.

3 Essentials for success

- Good knowledge of classical numerical methods, particularly for hyperbolic equations and/or incompressible equations.
- Solid knowledge on C++, fortran or python language, software version control with Git and software design.
- Basic knowledge of the analysis of partial differential equations, in particular linear space, hyperbolic equations and/or incompressible equations.
- An interest in environmental issues, mathematical formalism and numerical simulations.
- Good ability to write scientific reports.
- Language: English (spoken, written and read).

References

[1] T. B. Benjamin, J. L. Bona, and J. J. Mahony. Model equations for long waves in nonlinear dispersive systems. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 272(1220):47–78, 2024/01/23 1997.

[2] C. Escalante, E. D. Fernández-Nieto, J. Garres-Díaz, T. Morales de Luna, and Y. Penel. Non-hydrostatic layer-averaged approximation of euler system with enhanced dispersion properties. *Computational and Applied Mathematics*, 42(4):177, May 2023.

[3] E. D. Fernández-Nieto, M. Parisot, Y. Penel, and J. Sainte-Marie. A hierarchy of dispersive layer-averaged approximations of Euler equations for free surface flows. *Communications in Mathematical Sciences*, 16(5):1169–1202, 2018.

[4] A. E. Green and P. M. Naghdi. A derivation of equations for wave propagation in water of variable depth. *Journal of Fluid Mechanics*, 78(2):237–246, 1976.

[5] J. Guermond, P. Minev, and J. Shen. An overview of projection methods for incompressible flows. *Computer Methods in Applied Mechanics and Engineering*, 195(44):6011 – 6045, 2006.

[6] J. M.-K. Hong, J. Wu, and J.-M. Yuan. The generalized buckley-leverett and the regularized buckley-leverett equations. *Journal of Mathematical Physics*, 53(5):053701, 05 2012.

[7] D. Lannes. Modeling shallow water waves. *Nonlinearity*, 33(5):R1–R57, mar 2020.

[8] H. Nishikawa. A first-order system approach for diffusion equation. i: Second-order residual-distribution schemes. *Journal of Computational Physics*, 227(1):315–352, 2007.

[9] S. Noelle, M. Parisot, and T. Tscherpel. A class of boundary conditions for time-discrete Green-Naghdi equations with bathymetry. *SIAM Journal on Numerical Analysis*, 2022.

[10] M. Parisot. Entropy-satisfying scheme for a hierarchy of dispersive reduced models of free surface flow. *International Journal for Numerical Methods in Fluids*, 91(10):509–531, 2019.

[11] M. Parisot. *Some modeling and computational aspects of water waves action*. Habilitation à diriger des recherches, Université de Bordeaux, Nov. 2024.

[12] M. Parisot. Thick interface coupling technique for weakly dispersive models of waves. *ESAIM: M2AN*, 58(4):1497–1522, 2024.

[13] M. Parisot. Derivation of weakly hydrodynamic models in the Dupuit–Forchheimer regime. *Advances in Water Resources*, 199:104933, 2025.

[14] D. H. Peregrine. Long waves on a beach. *Journal of Fluid Mechanics*, 27(04):815–827, 1967.